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MATHEMATICS RESEARCH CENTER

CONVERGENCE OF PERIODIC WAVETRAINS IN THE LIMIT OF LARGE WAVELENGTH

Jerry L. Bona*

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ABSTRACT

The Korteweg-de Vries equation was originally derived as a model for unidirectional propagation of water waves. This equation possesses a special class of traveling-wave solutions corresponding to surface solitary waves. It also has permanent-wave solutions which are periodic in space, the so-called cnoidal waves. A classical observation of Korteweg and de Vries was that the solitary wave is obtained as a certain limit of cnoidal wavetrains.

This result is extended here, in the context of the Korteweg-de Vries equation. It is demonstrated that a general class of solutions of the Korteweg-de Vries equation is obtained as limiting forms of periodic solutions, as the period becomes large.

AMS (MOS) Subject Classifications: 35B10, 35B30, 35B45, 35Q20, 76B15, 76B25

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SIGNIFICANCE AND EXPLANATION

The Korteweg-de Vries equation was originally derived as a model for unidirectional propagation of water waves. This equation possesses a special class of traveling-wave solutions which are periodic in space, the so-called cnoidal waves. A classical observation of Korteweg and de Vries was that the solitary wave is obtained as a certain limit of cnoidal wavetrains.

This result is extended here, in the context of the Korteweg-de Vries equation. It is demonstrated that a general class of solutions of the Korteweg-de Vries equation is obtained as limiting forms of periodic solutions, as the period becomes large.

In practice, the KdV equation (as the Korteweg-de Vries equation will be referred to henceforth) is often used in the context of an initial-value problem. That is, the state of the medium to which the equation pertains is supposed known at a given instant of time. Inquiry is then focused on the subsequent evolution of the medium. The corresponding specification for the KdV equation is to give the value of the dependent variable $\eta(x,t)$ at, say, $t = 0$, and for all real x. As η typically represents a relative displacement of the medium in question, $\eta(x,0)$ is referred to as an initial wave profile. In case the initial wave profile is a smooth function decaying to 0 at $\pm\infty$, the resulting system is designated the pure initial-value problem for the KdV equation. An alternative, which has also been used in practical studies relating to the KdV equation, is to have the initial wave profile be a given periodic function. This latter situation is called the periodic initial-value problem for the KdV equation. The main result of this paper may be formulated somewhat more explicitly as saying that a certain class of solutions of the periodic initial-value problem converges to solutions of the pure initial-value problem in a particular limit of indefinitely large period.

Such a result has several consequences. One concerns the numerical solution of the pure initial-value problem for the KdV equation using a computer code for the periodic initial-value problem. This is a commonly-used strategy for various wave equations which is convenient for certain technical reasons. For example, the periodic problem is much easier to analyze. And it has the advantage of not requiring the specification of bounday conditions at some finite point, which a direct attack on the pure initial-value problem would necessarily entail, due to the truncation of the infinite domain. The theory developed here shows that, in principle, this strategy is not ill-founded, at least for initial data having bounded support, or decaying to 0 sufficiently rapidly at $\pm\infty$.

Another interesting suggestion concerns the inverse-scattering theory for the KdV equation. This theory pertains, in somewhat different forms, to both the pure initial-value problem and the periodic initial-value problem. Many fruitful studies have been undertaken to elucidate these ideas, and the associated Hamiltonian structure, in the context of the KdV and other wave equations. The results of sections 2 and 3 give credance to the suggestion that the inverse-scattering theory for the periodic and pure initial-value problems are related, in the limit of large wavelength.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

CONVERGENCE OF PERIODIC WAVETRAINS
IN THE LIMIT OF LARGE WAVELENGTH

Jerry L. Bona*

This paper is dedicated, fondly and with admiration,
to Professor L. J. F. Broer on his 65th birthday.

1. Introduction. In 1895, Korteweg and de Vries [1] derived the nonlinear partial differential equation

$$\eta_t + \eta_x + \eta\eta_x + \eta_{xxx} = 0 \,, \tag{1}$$

as a model for waves propagating on the surface of water which is confined within a rectangular canal. In the form depicted in (1), the dependent variable η represents the vertical deviation of the free surface from its equilibrium position while the independent variables x and t are proportional, respectively, to distance measured in the direction of propagation, and to elapsed time. These variables are dimensionless, but unscaled. Of course a number of assumptions are involved in arriving at (1) as a model. These will play no explicit role in the theory developed here, and we may therefore safely refer the reader to the articles of Benjamin [2], Benjamin et. al. [3], Broer [4, 5, 6], Broer et. al. [7], and the book of Whitham [8], for detailed commentary on these issues.

An especially interesting aspect of this equation is a one-parameter family of traveling-wave solutions,

$$\eta(x,t) = s_C(x,t) = 3C\cdot\mathrm{sech}^2\{\tfrac{1}{2} C^{1/2}[x - (C + 1)t]\} \,, \tag{2}$$

which, for $0 < C << 1$, correspond generally to the solitary wave first observed in the field by Scott Russell [9]. Also contained in the original paper of Korteweg and de Vries was an analysis of spatially-periodic wavetrains of permanent form, the so-called cnoidal waves. A class of these has the form

$$\eta(x,t) = 3C\cdot cn^2\{\tfrac{1}{2} (C - d)^{1/2}[x - (C + d + 1)t];k\} \,, \tag{3}$$

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where $C > 0 > d$, $k^2 = C/C - d$, and cn is the Jacobian elliptic function. This solution is periodic of period

$$p = \frac{4K(k)}{(C - d)^{1/2}} ,$$

where $K(k)$ is the complete elliptic integral of the first kind. Since both the cnoidal wavetrains (3) and the solitary-wave solutions (2) of equation (1) were obtained in closed form, it was a simple matter for Korteweg and de Vries to determine that their solitary-wave solutions were an appropriate limit of cnoidal waves as the period p approaches infinity. (As $d \uparrow 0$, $k \uparrow 1$, the elliptic function cn degenerates into the hyperbolic function sech, uniformly on bounded subsets of the real line $\mathbb{R}$, and the period p becomes infinite.)

Since the pioneering work of Korteweg and de Vries, an analogous theorem has been established in the context of solitary-wave solutions of the two-dimensional Euler equations (cf. Amick and Toland [10] and the references included therein). Results along the same lines have also been obtained for solitary waves arising in other contexts as well (cf. Benjamin, Bona and Bose [11], Bona and Bose [12], Bona, Bose and Turner [13], and Turner [14]).

In the next section, a simple criterion is presented that guarantees the convergence of spatially-periodic waves in the limit of large period. This result is not restricted to waves of permanent form. In section three, it is shown that a general class of solutions of the KdV equation, as (1) will be referred to henceforth, may be obtained as limits of periodic solutions. In the last section, some implications of the theory are briefly considered. Finally, in the appendix, a proof is outlined of the main result in section two.

2. A Criterion for Convergence of Periodic Wavetrains. To state the result in view here, some preliminary notions are needed. For any interval I in $\mathbb{R}$, let $L_2(I)$ denote the Hilbert space of real-valued measurable and square-integrable functions defined on I. For a non-negative integer k, the Hilbert space $H^k(I)$ is the linear subspace of

$L_2(I)$ whose first k (distributional) derivatives lie in $L_2(I)$. The norm of a function f in $H^k(I)$ is

$$\|f\|_{H^k(I)} = \left\{\sum_{j=0}^{k} \|f^{(j)}\|^2_{L_2(I)}\right\}^{1/2}, \tag{4}$$

where

$$\|g\|_{L_2(I)} = \left\{\int_I g(x)^2 dx\right\}^{1/2}.$$

Of course $H^0(I) = L_2(I)$.

We shall also have use for periodic versions of these spaces. The letter p will be systematically used to denote a half-period of a periodic function. For k as above, H_p^k will denote the class of functions $f : \mathbf{R} \to \mathbf{R}$, periodic of period $2p$, and such that f and its first k derivatives are square-integrable over the interval $[-p,p]$. The norm of a function in H_p^k is just that of $H^k([-p,p])$.

To follow the evolution in time of solutions of the KdV equation, the spaces $C(0,T;X)$ are introduced. If X is any Banach space, and J a closed real interval, then $C(J;X)$ is the collection of bounded continuous functions $u : J \to X$. This space carries a Banach-space structure, induced by the norm

$$\|u\|_{C(J;X)} = \sup_{t \in J} \|u(t)\|_X ,$$

where $\| \ \|_X$ denotes the norm defined on X. For $T > 0$, the notation $C(0,T;X)$ is just an abbreviation for $C(J;X)$ where $J = [0,T]$.

The KdV equation has a satisfactory theory of existence, uniqueness and continuous dependence corresponding to initial data in $H^k(\mathbf{R})$ or in H_p^k, for $k \geq 2$. In physical terms, it is imagined that the state of the system is known completely at a given instant of time, and that inquiry is made about the subsequent development of the system. Thus equation (1) is posed, subject to the auxillary condition

$$\eta(x,0) = f(x) ,$$

for $x \in \mathbf{R}$. The results of Bona and Smith [15] or Kato [16] imply the following.

PROPOSITION 1. Let $f \in H^k(\mathbb{R})$, where $k \geq 2$. Then for any $T > 0$, there is a unique solution η of (1) in $C(0,T;H^k(\mathbb{R}))$ satisfying the initial condition $\eta(x,0) = f(x)$, for all x in $\mathbb{R}$. The correspondence $f \to \eta$ is continuous from $H^k(\mathbb{R})$ to $C(0,T;H^k(\mathbb{R}))$. Moreover, for each integer j in the range $[0,k]$ there is a polynomial P_j with positive coefficients such that, for all $t \geq 0$,

$$\|\eta(\cdot,t)\|_{H^j(\mathbb{R})} \leq P_j(\|f\|_{H^j(\mathbb{R})}) . \tag{5}$$

Exactly the same results hold good if, for given $p > 0$, H^k_p replaces $H^k(\mathbb{R})$ throughout. The same polynomial P_j intervenes in the bound corresponding to (5), independently of p

Remarks. If $k \geq 4$, then the solution, guaranteed by the above proposition to exist, is classical. That is, η and all its partial derivatives appearing in the KdV equation are bounded and continuous functions of x and t, and the equation is satisfied pointwise everywhere. Otherwise the solution is interpreted in the distributional sense.

The situation where the initial data lies in $H^k(\mathbb{R})$, and so decays to zero at $\pm\infty$, will be referred to as the pure initial-value problem for KdV. The alternate specification, namely f in H^k_p for some $p > 0$, will be designated the periodic initial-value problem for KdV. Both of these types of problem have been used in applications of the KdV equation.

Here is the promised result concerning convergence of periodic solutions of the KdV equation.

PROPOSITION 2. Let $k \geq 2$. Let $f \in H^k(\mathbb{R})$, and for $p \geq p_0$, let $f_p \in H^k_p$ be given. Suppose that

$$\|f - f_p\|_{H^k([-p,p])} \to 0 , \tag{6}$$

as $p \to \infty$. Let η and η_p be the solutions of the KdV equation corresponding to the initial data f and f_p, respectively, for all $p \geq p_0$. Then for any $T > 0$ and bounded interval I in $\mathbb{R}$,

$$\eta_p \to \eta, \quad \text{in} \quad C(0,T;H^{k-2}(I)) ,$$

as $p \to \infty$.

A proof of this result is outlined in the appendix. In the next section, simple constructions are presented which lead to situations in which (6) is satisfied.

3. Solutions of the KdV Equation as Limits of Periodic Wavetrains. Suppose there is provided a reasonably smooth initial wave profile f, say in $H^k(\mathbb{R})$ where $k \geq 2$. When can f be approximated, in the sense specified in (6), by smooth periodic functions? The answer is that this is always possible. So, because of proposition 2, the solutions of the KdV equation corresponding to the initial profile f, which are guaranteed to exist by virtue of proposition 1, are always limits of periodic solutions of the KdV equation.

Here we briefly sketch two different methods of approximating f by periodic wavetrains. Let $p_0 > 0$ be specified and let $\theta : \mathbb{R} \to [0,1]$ be an infinitely differentiable function with $\theta(x) \equiv 1$ for $|x| \leq p_0/2$ and $\theta(x) \equiv 0$ for $|x| \geq p_0$. For $p \geq p_0$, define f_p as follows:

$$f_p(x) = \begin{cases} f(x), & \text{for} \quad |x| \leq p - p_0 , \\ f(x)\theta(x - p + p_0), & \text{for} \quad p - p_0 \leq x \leq p , \\ f(x)\theta(x + p - p_0), & \text{for} \quad -p \leq x \leq p_0 - p , \end{cases} \tag{7}$$

and extend f_p to the rest of $\mathbb{R}$ by demanding it be periodic of period $2p$. Roughly speaking, f has been restricted to the interval $[-p,p]$ and then altered near the endpoints of this interval. The altered function vanishes, with all its derivatives up to order $k - 1$, at $\pm p$, and consequently it may be extended by periodicity without loss of smoothness. It is straightforward to check that (6) holds for $\{f_p\}_{p \geq p_0}$.

Here is another way of approximating f by periodic functions. First approximate f in $H^k(\mathbb{R})$ by infinitely differentiable functions with bounded support. (The support of a function g is the closure of $\{x : g(x) \neq 0\}$. The class of all smooth functions with bounded support will be denoted C_0^∞.) This latter approximation is always possible (cf. Lions and Magenes [17], chapter 1). So let $\{\phi_p\}_{p>0} \subseteq C_0^\infty$ have the properties that $\phi_p \to f$ in $H^k(\mathbb{R})$ and each ϕ_p has its support contained in $[-p,p]$. Define ψ_p by the formula

$$\psi_p(x) = \sum_{n=-\infty}^{\infty} \phi_p(x + 2np) \ . \tag{8}$$

Plainly ψ_p lies in H_p^k, for all $p > 0$. Condition (6) is easily verified in this case as well.

In particular, if f, itself, has bounded support, or if, for example, there are positive constants C and δ so that

$$|f^{(j)}(x)| \leq \frac{C}{(1 + |x|)^{1+\delta}} \ ,$$

for $0 \leq j \leq k$, then formula (8) may be applied with f replacing ϕ_p, for all $p > 0$. This yields a 2p-periodic function f_p, with

$$f_p(x) = \sum_{n=-\infty}^{\infty} f(x + 2np) \ , \tag{9}$$

which lies in H_p^k. Again, condition (6) is easily established.

The above discussion is formalized in the following proposition.

PROPOSITION 3. Let $f \in H^k(\mathbb{R})$, where $k \geq 2$. Let $T > 0$ be fixed and let $\eta \in C(0,T;H^k(\mathbb{R}))$ be the solution of the KdV equation with initial data f. Then, for all $p > 0$, there are periodic solutions η_p in H_p^k of the KdV equation such that, for each interval I in $\mathbb{R}$,

$$\eta_p \to \eta \quad \text{in} \quad C(0,T;H^{k-2}(I)) \ ,$$

as $p \to \infty$.

4. Some Consequences. The idea outlined above, and dealt with in more detail in the appendix, has at least two implications worthy of comment.

The first concerns the numerical solution of the pure initial-value problem for the KdV equation using a computer code for the periodic initial-value problem. This is a commonly-used strategy for various wave equations (cf. Meiss and Pereira [18] for a recent instance) which is convenient for certain technical reasons. For example, the periodic problem is much easier to analyze. And it has the advantage of not requiring the specification of boundary conditions at some finite point, which a direct attack on the pure initial-value problem would necessarily entail, due to the truncation of the infinite domain. The theory developed here shows that, in principle, this strategy is not ill-founded, at least for initial data having bounded support, or decaying to 0 sufficiently rapidly at $\pm\infty$. (For instance, a solitary-wave profile, as in (2), has, as a practical matter, bounded support.) For in such a case, if the period length $2p$ is taken a good deal larger than the support of the initial data f, then solving the periodic problem with initial data f_p, as in (9), yields results agreeing closely with the solution of the pure initial-value problem with data f.

To be useful in practice, one would need to supplement the above remarks with an effective estimate of the relationship between the time-interval $[0,T]$ and the period $2p$ over which the solution u_p of the periodic problem remains a faithful representation of the solution u of the pure initial-value problem. Without going into details, it appears that u_p will generally remain close to u at least on a time scale of order $(p - m)/\|f\|_{H^1}$, where it is supposed that the support of the initial data f lies in the interval $[-m,m]$. This conclusion subsists on several presumptions, and so should be viewed as conjectural.

One of the ingredients that point to the just-discussed estimate are some very simple facts that derive from the inverse-scattering theory for the KdV equation. This theory pertains, in somewhat different forms, to both the pure initial-value problem and the periodic initial-value problem. Many fruitful studies have been undertaken to elucidate these ideas, and the associated Hamiltonian structure, in the context of the KdV and other wave equations (cf. Broer [19], Deift and Trubowitz [20], McKean and van Moerbeke [21] and Trubowitz [22]). The results of sections 2 and 3 indicate that the inverse-scattering theory for the periodic and pure initial-value problems are related, in the limit of large wavelength. In fact, one may be a good deal more precise about this relationship, but the issue will not be explored here.

Finally, it is worth pointing out that the detailed structure of the KdV equation is not crucial to the conclusions established in propositions 2 and 3. Indeed, a purusal of the proof in the appendix will convince the reader that a similar theory may be established for a whole range of one-dimensional nonlinear wave equations.

<u>Appendix</u>. A proof of proposition 2 is presented here. The proof calls upon some elementary functional analysis together with the theory for the KdV equation, as set forth in proposition 1.

For the purposes of the present section, some additional function classes are needed. Let X be any Banach space. *Let* $1 \leqslant q < \infty$ *and* let J be an open interval in $\mathbf{R}$. We denote by $L^q(J;X)$ the (equivalence classes of) measurable functions $u : J \to X$ such that

$$\|u\|_{L^q(J;X)} = \left\{\int_J \|u(t)\|_X^q dt\right\}^{1/q} < +\infty .$$

For $q = \infty$, $L^\infty(J;X)$ is the measurable functions $u : J \to X$ which are essentially bounded. The norm on this space is

$$\|u\|_{L^\infty(J;X)} = \underset{t \in J}{\text{essential sup}} \|u(t)\|_X .$$

These are all Banach spaces in their own right. Additionally, if $1 < q \leqslant \infty$, and X is reflexive, then $L^q(J;X)$ is the dual space of $L^r(J;X^*)$, where $r = q/(q-1)$. (If $q = \infty$, $r = 1$.) Hence, according to the general theory of Banach spaces, any closed ball of radius R, say, in $L^q(J;X)$ is compact for the weak-star topology on $L^q(J;X)$ induced by $L^r(J;X)$.

For J and X as above, $\mathcal{D}'(J;X)$ denotes the X-valued distributions defined on J. Formally,

$$\mathcal{D}'(J;X) = B(C_0^\infty(J);X) ,$$

where $C_0^\infty(J)$ is the set of infinitely-differentiable real-valued functions defined on J, and having compact support in J. And, $B(Y,Z)$ denotes the continuous linear mappings between the linear spaces Y and Z. If $u \in \mathcal{D}'(J;X)$, its distributional derivative is defined by

$$\frac{du}{dt}(\phi) = -u\left(\frac{d\phi}{dt}\right) ,$$

for all ϕ in $C_0^\infty(J)$. If $u \in L^q(J;X)$, then u corresponds to an element in $\mathcal{D}'(J;X)$ via the rule

$$u(\phi) = \int_J u(t)\phi(t)dt ,$$

for ϕ in $C_0^\infty(J)$. The integral is X-valued, and converges since ϕ has compact support. Thus one can always define the derivative of elements in $L^q(J;X)$, at least in the foregoing weak sense. For details concerning these spaces, the reader is urged to consult Lions and Magenes [17] or Lions [23].

Let I be a fixed bounded interval in $\mathbb{R}$ and let $T > 0$. Let k and m be integers, with $k > m$. Let

$$W(I) = \{u \in L^2(0,T;H^k(I)) : \partial_t u \in L^2(0,T;H^m(I))\} .$$

$W(I)$ is equipped with the norm

$$\|u\|_{W(I)} = \|u\|_{L^2(0,T;H^k(I))} + \|\partial_t u\|_{L^2(0,T;H^m(I))} .$$

(Here the abbreviation $L^q(0,T;X)$ for $L^q((0,T);X)$ has been introduced.) It follows from a general compactness lemma (cf. Lions [23], ch. 1, section 5) that $W(I)$ is compactly imbedded in $L^2(0,T;H^{k-1}(I))$. Define

$$Y(I) = \{u \in L^2(0,T;H^{k-1}(I)) : \partial_t u \in L^2(0,T;H^m(I))\} .$$

$Y(I)$ is equipped with a norm analogous to $W(I)$. It is again a general result (cf. Lions and Magenes [17], ch. 1, section 3.1) that $Y(I)$ is continuously imbedded in $C(0,T;H^{(k+m-1)/2}(I))$. In particular, for $m = k - 3$, we have

$$W_k(I) = \{u \in L^2(0,T;H^k(I)) : \partial_t u \in L^2(0,T;H^{k-3}(I))\}$$

compactly imbedded in $C(0,T;H^{k-2}(I))$.

With these preliminary remarks in hand, the proof is now straightforward. Let f_p, for $p \geq p_0$, and f be given, as in the statement of proposition 2. Let u_p, for $p \geq p_0$, and u be the solutions of the KdV equation corresponding to the initial data f_p, for $p \geq p_0$, and f, respectively. Fix $T > 0$ and a bounded interval I in $\mathbf{R}$. First note that, because of the a priori bounds stated in (5) of proposition 1, and because of the assumption (6), for $0 \leq j \leq k$,

$$\|u_p(\cdot,t)\|_{H^j(I)} \leq \|u_p(\cdot,t)\|_{H^j_p} \leq P_j(\|f_p\|_{H^j_p})$$

$$\leq P_j(\|f\|_{H^j(\mathbf{R})} + 1) = M_j ,$$

say, for all $t \geq 0$, provided p is sufficiently large. Thus, for $0 \leq j \leq k$,

$$\limsup_{p\to\infty} \|u_p(\cdot,t)\|_{H^j(I)} \leq M_j , \tag{10}$$

independently of the interval I. In particular, using the differential equation (1), and elementary estimates, we deduce that

$$\limsup_{p\to\infty} \|\partial_t u_p(\cdot,t)\|_{H^{k-3}(I)} \leq M_{k-2} + cM_{k-2}^2 + M_k = N_k , \tag{11}$$

say. Here c is a constant depending only on k.

Now let $\{p_j\}_{j=1}^\infty$ be any sequence of half-periods, with $p_j \to \infty$ as $j \to \infty$. Let $u_j = u_{p_j}$. Let also $\{I_\ell\}_{\ell=1}^\infty$ be an increasing sequence of closed bounded intervals in $\mathbf{R}$ with $\mathbf{R} = \cup_\ell I_\ell$. Then because of (10) and (11),

$\{u_j\}$ is a bounded sequence in $W_k(I_\ell)$,

$\{u_j\}$ is a bounded sequence in $L^\infty(0,T;H^k(I_\ell))$,

$\{\partial_t u_j\}$ is a bounded sequence in $L^\infty(0,T;H^{k-3}(I_\ell))$,

for all ℓ. Hence, for each ℓ, $\{u_j\}$ lies in a compact subset of $C(0,T;H^{k-2}(I_\ell))$ and of $L^\infty(0,T;H^k(I_\ell))$ with the weak-star topology. Similarly, $\{\partial_t u_j\}$ lies in a compact subset of $L^\infty(0,T;H^{k-3}(I_\ell))$ with the weak-star topology. It follows, by a diagonalization argument, that there is a subsequence $\{p_r\}_{r=1}^\infty$ of $\{p_j\}_{j=1}^\infty$ and a function U, which lies in $L^\infty(0,T;H^k(I_\ell))$ for all ℓ, such that

$$\begin{aligned} &\text{i)} \quad u_r = u_{p_r} \to U, \quad \text{weak-star in } L^\infty(0,T;H^k(I_\ell)), \\ &\text{ii)} \quad \partial_t u_r \to \partial_t U, \quad \text{weak-star in } L^\infty(0,T;H^{k-3}(I_\ell)), \\ &\text{iii)} \quad u_r \to U, \quad \text{strongly in } C(0,T;H^{k-2}(I_\ell)), \end{aligned} \tag{12}$$

as $r \to \infty$, for each ℓ. (Note for (ii) that one infers first that $\partial_t u_r \to V$, for some V, and then deduces that $V = \partial_t U$ since (i) certainly implies $\partial_t u_r \to \partial_t U$ at least for $\mathcal{D}'(0,T;H^k(I_\ell))$. It is straightforward to show that U is a (distributional) solution of the KdV equation, and that $U(x,0) \equiv f(x)$.

Let $L > 0$ be given. Then, since weak-star convergence is lower semicontinuous relative to the norm in the relevant space,

$$\|U\|_{L^\infty(0,T;H^k([-L,L]))} \leq \limsup_{r\to\infty} \|u_r\|_{L^\infty(0,T;H^k([-L,L]))} \leq M_k .$$

The constant M_k does not depend on L. So the monotone convergence theorem implies that, for almost every t in $[0,T]$, $U(\cdot,t)$ lies in $H^k(\mathbb{R})$, and that

$$\|U(\cdot,t)\|_{H^k(R)} \leq M_k .$$

Thus U is seen to lie in $L^\infty(0,T;H^k(\mathbb{R}))$. In consequence of the uniqueness theorem in proposition 1 (which continues to hold in this somewhat wider function class), $U = u$, almost everywhere in $\mathbb{R} \times [0,T]$. It now follows from (12, iii), and that fact that $I \subseteq I_\ell$, for large ℓ, that as $r \to \infty$,

$$u_r \to u, \quad \text{in } C(0,T;H^{k-2}(I)) . \tag{13}$$

It has been shown that an arbitrary sequence of half-periods $\{p_j\}$ with $p_j \to \infty$, as $j \to \infty$, admits a subsequence for which the conclusion (13) is valid. It follows immediately that

$$u_p \to u, \quad \text{in} \quad C(0,T;H^{k-2}(I)) \ ,$$

as $p \to \infty$. *Since* I *and* T were arbitrary, the proposition is established.

Its worth remarking, finally, that the conclusion of proposition 2 can be strengthened. However, the proof becomes more technical. Since the basic thrust of the proposition is the same, whether the conclusion be sharply stated or not, the development contained herein has been preferred.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The Korteweg-de Vries equation was originally derived as a model for unidirectional propagation of water waves. This equation possesses a special class of traveling-wave solutions corresponding to surface solitary waves. It also has permanent-wave solutions which are periodic in space, the so-called cnoidal waves. A classical observation of Korteweg and de Vries was that the solitary wave is obtained as a certain limit of cnoidal wavetrains.

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ABSTRACT (continued)

This result is extended here, in the context of the Korteweg-de Vries equation. It is demonstrated that a general class of solutions of the Korteweg-de Vries equation is obtained as limiting forms of periodic solutions, as the period becomes large.

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